SEM-4 (MATHEMATICS GENERAL)
(ANUP RAY)

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Proof. (Th. 2)

Let H be normal in G then $aH = Ha \; \forall \; a \in G$
Let $h \in H$, $x \in G$ be any elements, then

$hx \in Hx = xH$

$\Rightarrow hx = xh_1$ for some $h_1 \in H$

$\Rightarrow x^{-1}hx = h_1 \in H$

Thus the condition is necessary.
Conversely, let $a \in G$ be any element, then

$a^{-1}ha \in H$ for all $h \in H$

$\Rightarrow a(a^{-1}ha) \in aH$ for all $h \in H$

$\Rightarrow ha \in aH$ for all $h \in H$

$\Rightarrow Ha \subseteq aH$ --- (1)

Let $b = a^{-1}$ and $b \in G$

$b^{-1}hb \in H$, for all $h \in H$

$\Rightarrow aha^{-1} \in H$ for all $h \in H$

$\Rightarrow (aha^{-1})a \in Ha$ for all $h \in H$

$\Rightarrow ah \in Ha$ for all $h \in H$

$\Rightarrow aH \subseteq Ha$ --- (2)

From (1) & (2) we have $Ha = aH$

Thus H is normal.

Th: 3 The intersection of any two normal sub-groups of a group G is a normal sub-group.

Proof: Let H and K be two normal subgroups of G.
Then H and K are obviously are subgroups of G
Thus $H \cap K$ is a subgroup of G.

Hence $a \in H \cap K \Rightarrow a \in H$ and $a \in K$.

Since H is a normal sub-group, so $xax^{-1} \in H \; \forall \; x \in G$, $a \in H$.

Again K is a normal subgroup, so $xax^{-1} \in K$ $\forall \; x \in G$, $a \in K$.

Thus $xax^{-1} \in H$ and $xax^{-1} \in K \Rightarrow xax^{-1} \in H \cap K$.

$\therefore$ $x \in G$, $a \in H \cap K \Rightarrow xax^{-1} \in H \cap K$

Hence $H \cap K$ is a normal subgroup of G.

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Note: Union of two normal subgroups of a group G may not be a normal subgroup, because the union of two subgroups may not be a subgroup of G.

Th: 4 A subgroup H of a group G is normal subgroup of G if and only if product of two right cosets of H in G is again a right coset of H in G.

Proof: ~~A subgro~~ Let H be a normal subgroup of G.
Let Ha and Hb be any two right cosets of H in G.
Then $(Ha)(Hb) = H(aH)b = H(Ha)b$
$= HHab = Hab, \; ab \in G$
Conversely, we are given that product of any two right cosets of H in G is again a right coset.
We are to prove that H is normal.
Let $x \in G$ be any element. Then Hx and Hx^{-1} are two right cosets of H in G. Thus ~~H~~ $HxHx^{-1}$ is also a right cosets of H in G.
We claim $HxHx^{-1} = He$
Now ~~e~~ $exex^{-1} \in HxHx^{-1} \Rightarrow e \in HxHx^{-1}$
Also $e \in H$. Thus H and $HxHx^{-1}$ are two right cosets having one element common.
Recalling the properties of equivalence classes we know that two right cosets are either equal or have no element in common. Thus, (as e is common element) $H = HxHx^{-1}$
Now $hxh_1x^{-1} \in HxHx^{-1} \;\forall h, h_1 \in H, x \in G$
$\Rightarrow hxh_1x^{-1} \in H \;\forall h, h_1 \in H, x \in G$
$\Rightarrow h^{-1}(hxh_1x^{-1}) \in h^{-1}H$
$\Rightarrow xh_1x^{-1} \in H \;\forall h_1 \in H, x \in G$
$\Rightarrow$ H is a normal.